

A new generalization of binomial coefficients

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Abstract

Let t be a fixed parameter and x some indeterminate. We give some properties of the generalized binomial coefficients $\langle x \rangle_k$ inductively defined by $k/x \langle x \rangle_k = t \langle x-1 \rangle_{k-1} + (1-t) \langle x-2 \rangle_{k-2}$.

1 Definition

There are many generalizations of binomial coefficients, the most elementary of which are the Gaussian polynomials. In this note, we shall present another one-parameter extension, presumably new, encountered in the study of the symmetric groups [4]. This application will be described at the end.

Let t be a fixed parameter and x some indeterminate. For any positive integer k , we define a function $\langle x \rangle_k$ inductively by $\langle x \rangle_k = 0$ for $k < 0$, $\langle x \rangle_0 = 1$ and

$$\frac{k}{x} \langle x \rangle_k = t \langle x-1 \rangle_{k-1} + (1-t) \langle x-2 \rangle_{k-2}.$$

Then $\langle x \rangle_k$ is a polynomial with degree k in x and in t . First values are given by

$$\begin{aligned} \langle x \rangle_1 &= tx, & \langle x \rangle_2 &= t^2 \binom{x}{2} + (1-t) \frac{x}{2}, \\ \langle x \rangle_3 &= t^3 \binom{x}{3} + t(1-t) \binom{x}{2} - \frac{1}{3} t(1-t)x, \\ \langle x \rangle_4 &= t^4 \binom{x}{4} + \frac{3}{2} t^2(1-t) \binom{x}{3} - \frac{1}{12} (1-t)(8t^2 + 3t - 3) \binom{x}{2} + \frac{1}{8} (1-t^2)(2t-1)x. \end{aligned}$$

For k odd, it is obvious that $\langle x \rangle_k$ is divisible by t .

We have easily

$$\langle x \rangle_k = t^k \binom{x}{k} + (1-t)xP(x, t),$$

with P a polynomial of degree $k-2$ in x and t . For $t=1$ we recover the classical binomial product

$$\binom{x}{k} = \frac{1}{k!} \prod_{i=1}^k (x-i+1),$$

and when x is some positive integer n , the binomial coefficient $\binom{n}{k}$.

In this paper we shall present some notable properties of the generalized binomial coefficients $\langle x \rangle_k$, including a generating function, a Chu-Vandermonde identity and an explicit formula. The referee has suggested that it would be interesting to obtain a q -analogue of our results, using q -shifted factorials instead of ordinary ones, in the same way than [5] has been generalized by [3].

2 Generating function

We consider the series

$$G(u) = 1 + tu + (1-t) \sum_{k \geq 0} \frac{(-u)^{k+2}}{(k+2)!} \prod_{i=0}^k (k-i+1+(t-1)i),$$

$$H(u) = 1 + \sum_{k \geq 1} \frac{(-u)^k}{k!} \prod_{i=1}^k (k-i+1+(t-1)i).$$

We have

$$\frac{d}{du} G(u) = t + (1-t)u H(u).$$

Theorem 1. *The series $G(u)$ and $H(u)$ are mutually inverse.*

Proof. Krattenthaler has pointed out that the statement is a consequence of Rothe identity [1, 2]

$$\sum_{k=0}^n \frac{A}{A+Bk} \binom{A+Bk}{k} \binom{C-Bk}{n-k} = \binom{A+C}{n}.$$

Actually if we denote

$$X_k = -\frac{k+1}{t-2}, \quad Y_k = X_{k-2} = -\frac{k-1}{t-2},$$

we have

$$G(u) = 1 + tu + \frac{1-t}{2-t} \sum_{k \geq 2} u^k (t-2)^k \frac{1}{Y_k+1} \binom{Y_k+1}{k},$$

$$= u + \frac{1-t}{2-t} \sum_{k \geq 0} u^k (t-2)^k \frac{1}{Y_k+1} \binom{Y_k+1}{k},$$

$$H(u) = \sum_{k \geq 0} u^k (t-2)^k \binom{X_k-1}{k}.$$

Rothe identity, written for

$$A = 1 + \frac{1}{t-2}, \quad B = -\frac{1}{t-2}, \quad C = -1 - \frac{n+1}{t-2},$$

yields

$$\frac{1-t}{2-t} \sum_{k=0}^n \frac{1}{Y_k+1} \binom{Y_k+1}{k} \binom{X_{n-k}-1}{n-k} = \binom{-n/(t-2)}{n} = \binom{X_{n-1}}{n},$$

since $X_{n-1} = -n/(t-2)$. Therefore for $n \neq 0$ the coefficient of u^n in $H(u)G(u)$ is

$$(t-2)^{n-1} \binom{X_{n-1}-1}{n-1} + (t-2)^n \binom{X_{n-1}}{n} = (t-2)^{n-1} \binom{X_{n-1}-1}{n-1} \left(1 + (t-2) \frac{X_{n-1}}{n}\right) = 0.$$

□

Corollary 1. *The series $G(u)$ is the unique solution of $G(0) = 1$ and*

$$\frac{d}{du}G(u) = t + (1-t) \frac{u}{G(u)}.$$

Theorem 2. *The generating function of the numbers $\langle x \rangle_k$ is given by*

$$\sum_{k \geq 0} \langle x \rangle_k u^k = (G(u))^x.$$

Proof. If we write

$$\mathcal{F}(u; x) = \sum_{k \geq 0} \langle x \rangle_k u^k,$$

the definition of $\langle x \rangle_k$ yields

$$\frac{1}{x} \frac{d}{du} \mathcal{F}(u; x) = t \mathcal{F}(u; x-1) + (1-t) u \mathcal{F}(u; x-2).$$

Inspired by the $t = 1$ case which is the classical binomial formula $\sum_{k \geq 0} \binom{x}{k} u^k = (1+u)^x$, we may look for a generating function of the form $\mathcal{F}(u; x) = (F(u))^x$. Then we have

$$\frac{d}{du} F(u) = t + (1-t) \frac{u}{F(u)}.$$

We apply Corollary 1. □

Corollary 2. For $k \geq 1$ we have

$$\left\langle \begin{matrix} -1 \\ k \end{matrix} \right\rangle = \frac{(-1)^k}{k!} \prod_{i=1}^k (k - i + 1 + (t - 1)i).$$

Proof. Consequence of $\mathcal{F}(u; -1) = H(u)$. □

Corollary 3. We have the generalized Chu-Vandermonde formula

$$\sum_{i=0}^k \left\langle \begin{matrix} x \\ i \end{matrix} \right\rangle \left\langle \begin{matrix} y \\ k-i \end{matrix} \right\rangle = \left\langle \begin{matrix} x+y \\ k \end{matrix} \right\rangle.$$

Proof. Standard consequence of $\mathcal{F}(u; x+y) = \mathcal{F}(u; x)\mathcal{F}(u; y)$. □

Corollary 4. A variant of the generalized Chu-Vandermonde formula is given by

$$\sum_{i=0}^k i \left\langle \begin{matrix} x \\ i \end{matrix} \right\rangle \left\langle \begin{matrix} y \\ k-i \end{matrix} \right\rangle = \frac{kx}{x+y} \left\langle \begin{matrix} x+y \\ k \end{matrix} \right\rangle.$$

Proof. By definition the left-hand side is given by

$$x \sum_{i=0}^k \left\langle \begin{matrix} y \\ k-i \end{matrix} \right\rangle \left(t \left\langle \begin{matrix} x-1 \\ i-1 \end{matrix} \right\rangle + (1-t) \left\langle \begin{matrix} x-2 \\ i-2 \end{matrix} \right\rangle \right).$$

By the generalized Chu-Vandermonde formula, it can be written as

$$xt \left\langle \begin{matrix} x+y-1 \\ k-1 \end{matrix} \right\rangle + x(1-t) \left\langle \begin{matrix} x+y-2 \\ k-2 \end{matrix} \right\rangle = x \frac{k}{x+y} \left\langle \begin{matrix} x+y \\ k \end{matrix} \right\rangle.$$

□

Remark. When $m \neq 0, 1$ we do not know any such simple expression for

$$\sum_{i=0}^k \binom{i}{m} \left\langle \begin{matrix} x \\ i \end{matrix} \right\rangle \left\langle \begin{matrix} y \\ k-i \end{matrix} \right\rangle.$$

Corollary 5. For $n \geq 2$ we have

$$\sum_{i=0}^n i \left\langle \begin{matrix} n \\ n-i \end{matrix} \right\rangle \left\langle \begin{matrix} -1 \\ i-1 \end{matrix} \right\rangle = 0.$$

Proof. Since we have

$$\sum_{i=0}^n i \left\langle \begin{matrix} n \\ n-i \end{matrix} \right\rangle \left\langle \begin{matrix} -1 \\ i-1 \end{matrix} \right\rangle = \sum_{i=0}^{n-1} (i+1) \left\langle \begin{matrix} n \\ n-i-1 \end{matrix} \right\rangle \left\langle \begin{matrix} -1 \\ i \end{matrix} \right\rangle,$$

the property follows from Corollaries 3 and 4, written with $x = -1$, $y = n$, $k = n - 1$. Actually in that case $kx/(x+y) + 1 = 0$. □

Looking for the contributions to u^m in $(G(u))^{x-1}G(u)$ we have the following generalization of Pascal's recurrence formula

$$\left\langle \begin{matrix} x \\ m \end{matrix} \right\rangle - \left\langle \begin{matrix} x-1 \\ m \end{matrix} \right\rangle = t \left\langle \begin{matrix} x-1 \\ m-1 \end{matrix} \right\rangle + (1-t) \sum_{k=0}^{m-2} \frac{(-1)^k}{(k+2)!} \left\langle \begin{matrix} x-1 \\ m-k-2 \end{matrix} \right\rangle \prod_{i=0}^k (k-i+1+(t-1)i).$$

Similarly with $(G(u))^{x+1}H(u)$ we get

$$\left\langle \begin{matrix} x \\ m \end{matrix} \right\rangle - \left\langle \begin{matrix} x+1 \\ m \end{matrix} \right\rangle = \sum_{k=1}^m \frac{(-1)^k}{k!} \left\langle \begin{matrix} x+1 \\ m-k \end{matrix} \right\rangle \prod_{i=1}^k (k-i+1+(t-1)i).$$

3 New properties

Let n be a positive integer. The previous properties of $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ are very similar to those of the classical binomial coefficient $\binom{n}{k}$. However some big differences must be emphasized.

Firstly $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ and $\left\langle \begin{matrix} n \\ n-k \end{matrix} \right\rangle$ are not equal. In particular $\left\langle \begin{matrix} n \\ n \end{matrix} \right\rangle \neq 1$. By definition we have

$$\left\langle \begin{matrix} n \\ n \end{matrix} \right\rangle = t \left\langle \begin{matrix} n-1 \\ n-1 \end{matrix} \right\rangle + (1-t) \left\langle \begin{matrix} n-2 \\ n-2 \end{matrix} \right\rangle.$$

which implies by induction

$$\left\langle \begin{matrix} n \\ n \end{matrix} \right\rangle = 1 + (t-1) \left\langle \begin{matrix} n-1 \\ n-1 \end{matrix} \right\rangle,$$

and

$$\left\langle \begin{matrix} n \\ n \end{matrix} \right\rangle = \frac{1 - (t-1)^{n+1}}{2-t}.$$

Secondly $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ is not zero for $k > n$, but divisible by $(1-t)$. Starting from the definition we have for $k \geq 2$,

$$\begin{aligned} k \left\langle \begin{matrix} 1 \\ k \end{matrix} \right\rangle &= (1-t) \left\langle \begin{matrix} -1 \\ k-2 \end{matrix} \right\rangle \\ &= (1-t) \frac{(-1)^k}{(k-2)!} \prod_{i=1}^{k-2} (k-i-1+(t-1)i). \end{aligned}$$

By induction we get

$$\begin{aligned} \binom{k}{2} \left\langle \begin{matrix} 2 \\ k \end{matrix} \right\rangle &= t(1-t) \left\langle \begin{matrix} -1 \\ k-3 \end{matrix} \right\rangle, \quad k \geq 3, \\ \binom{k}{3} \left\langle \begin{matrix} 3 \\ k \end{matrix} \right\rangle &= (1-t) \left(t^2 + \frac{k-1}{2}(1-t) \right) \left\langle \begin{matrix} -1 \\ k-4 \end{matrix} \right\rangle, \quad k \geq 4, \\ \binom{k}{4} \left\langle \begin{matrix} 4 \\ k \end{matrix} \right\rangle &= t(1-t) \left(t^2 + \frac{5k-8}{6}(1-t) \right) \left\langle \begin{matrix} -1 \\ k-5 \end{matrix} \right\rangle, \quad k \geq 5. \end{aligned}$$

More generally for $k > n$ the definition yields

$$\binom{k}{n} \langle n \rangle = (1-t)f_{n,k} \left\langle \begin{matrix} -1 \\ k-n-1 \end{matrix} \right\rangle,$$

where $f_{n,k}$ is a monic polynomial in t , inductively defined by $f_{1,k} = 1$, $f_{2,k} = t$ and

$$f_{n,k} = tf_{n-1,k-1} + (1-t)\frac{k-1}{n-1}f_{n-2,k-2}.$$

The coefficients $\langle n \rangle_k$ with $k > n$ may be written in terms of coefficients $\langle n \rangle_k$ with $k \leq n$. The simplest case is given below.

Proposition 1. *We have*

$$\frac{1}{1-t} \left\langle \begin{matrix} n \\ n+1 \end{matrix} \right\rangle = \sum_{i=0}^n \frac{i}{i+1} \left\langle \begin{matrix} n \\ n-i \end{matrix} \right\rangle \left\langle \begin{matrix} -1 \\ i-1 \end{matrix} \right\rangle.$$

Proof. Denoting the right-hand side by h_n , we must prove that $h_n = f_{n,n+1}/(n+1)$. Equivalently that h_n is inductively defined by

$$\frac{n+1}{n}h_n = th_{n-1} + (1-t)h_{n-2}.$$

In other words, that we have

$$\sum_{i=0}^n \frac{i}{i+1} \left(\frac{n+1}{n} \frac{n-i}{n-i} \left\langle \begin{matrix} n \\ n-i \end{matrix} \right\rangle - t \left\langle \begin{matrix} n-1 \\ n-i-1 \end{matrix} \right\rangle - (1-t) \left\langle \begin{matrix} n-2 \\ n-i-2 \end{matrix} \right\rangle \right) \left\langle \begin{matrix} -1 \\ i-1 \end{matrix} \right\rangle = 0.$$

By the definition this may be rewritten as

$$\sum_{i=0}^n \frac{i}{i+1} \left(\frac{n+1}{n-i} - 1 \right) \frac{n-i}{n} \left\langle \begin{matrix} n \\ n-i \end{matrix} \right\rangle \left\langle \begin{matrix} -1 \\ i-1 \end{matrix} \right\rangle = 0.$$

We apply Corollary 5. □

4 Binomial expansion

In this section we consider the expansion

$$\left\langle \begin{matrix} x \\ k \end{matrix} \right\rangle = t^k \binom{x}{k} + \sum_{i=1}^{k-1} c_i(k) \binom{x}{i}.$$

We give two methods for the evaluation of the coefficients $c_i(k)$, $1 \leq i \leq k-1$.

4.1 First method

The definition of $\langle x \rangle_k$ may be written as

$$\sum_{i=1}^k \frac{k}{i} c_i(k) \binom{x-1}{i-1} = t \sum_{i=1}^{k-1} c_i(k-1) \binom{x-1}{i} + (1-t) \sum_{i=1}^{k-2} c_i(k-2) \binom{x-2}{i}.$$

Using the classical identity

$$\binom{x}{i} = \sum_{m=0}^i (-1)^m \binom{x+1}{i-m},$$

and identifying the coefficients of $\binom{x-1}{i-1}$ on both sides, we obtain

$$\frac{k}{i} c_i(k) = t c_{i-1}(k-1) + (1-t) \sum_{m=0}^{k-i-1} (-1)^m c_{i+m-1}(k-2).$$

This relation may be used to get $c_i(k)$ inductively.

For $k \geq 6$ this recurrence yields

$$\begin{aligned} \langle x \rangle_k &= t^k \binom{x}{k} - \frac{k-1}{2} t^{k-2} (t-1) \binom{x}{k-1} \\ &\quad + \frac{k-2}{3} t^{k-4} (t-1) \left(t^2 + \frac{3(k-3)}{8} (t-1) \right) \binom{x}{k-2} \\ &\quad - \frac{k-3}{4} t^{k-6} (t-1) \left(t^4 + \frac{4k-13}{6} t^2 (t-1) + \frac{1}{6} \binom{k-4}{2} (t-1)^2 \right) \binom{x}{k-3} \\ &\quad + \frac{k-4}{5} t^{k-8} (t-1) \left(t^6 + \frac{65k-229}{72} t^4 (t-1) \right. \\ &\quad \quad \left. + \frac{5(2k-9)}{48} (k-5) t^2 (t-1)^2 + \frac{5}{64} \binom{k-5}{3} (t-1)^3 \right) \binom{x}{k-4} \\ &\quad - \frac{k-5}{6} t^{k-10} (t-1) \left(t^8 + \frac{66k-251}{60} t^6 (t-1) + \frac{85k^2-853k+2148}{240} t^4 (t-1)^2 \right. \\ &\quad \quad \left. + \frac{4k-23}{48} \binom{k-6}{2} t^2 (t-1)^3 + \frac{3}{80} \binom{k-6}{4} (t-1)^4 \right) \binom{x}{k-5} \\ &\quad + \dots \end{aligned}$$

It appears empirically that $c_{k-i}(k)$, $i \neq 0$, has the form

$$c_{k-i}(k) = (-1)^i \frac{k-i}{i+1} t^{k-2i} (t-1) \sum_{m=0}^{i-1} a_m(k, i) t^{2(i-m-1)} (t-1)^m,$$

with $a_m(k, i)$ a polynomial in k with degree m .

4.2 Second method

A much better expression of $c_i(k)$, $1 \leq i \leq k-1$, may be obtained by using the relation

$$\frac{1}{1-t} \binom{k}{n} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = f_{n,k} \left\langle \begin{matrix} -1 \\ k-n-1 \end{matrix} \right\rangle, \quad k \geq n+1. \quad (4.1)$$

For instance $c_1(k)$ is given by

$$\left\langle \begin{matrix} 1 \\ k \end{matrix} \right\rangle = c_1(k),$$

hence

$$\frac{1}{1-t} k c_1(k) = \frac{1}{1-t} k \left\langle \begin{matrix} 1 \\ k \end{matrix} \right\rangle = \left\langle \begin{matrix} -1 \\ k-2 \end{matrix} \right\rangle.$$

Similarly we have

$$\left\langle \begin{matrix} 2 \\ k \end{matrix} \right\rangle = c_2(k) + 2c_1(k),$$

which yields

$$\frac{1}{1-t} \binom{k}{2} c_2(k) = t \left\langle \begin{matrix} -1 \\ k-3 \end{matrix} \right\rangle - (k-1) \left\langle \begin{matrix} -1 \\ k-2 \end{matrix} \right\rangle.$$

This property is generalized by the following explicit formula.

Theorem 3. Let $f_{n,k}$ be the polynomial in t and k inductively defined by $f_{1,k} = 1$, $f_{2,k} = t$ and

$$f_{n,k} = t f_{n-1,k-1} + (1-t) \frac{k-1}{n-1} f_{n-2,k-2}.$$

We have

$$\left\langle \begin{matrix} x \\ k \end{matrix} \right\rangle = t^k \binom{x}{k} + \sum_{i=1}^{k-1} c_i(k) \binom{x}{i},$$

with $c_i(k)$ given by

$$\frac{1}{1-t} \binom{k}{i} c_i(k) = \sum_{m=1}^i (-1)^{i-m} f_{m,k} \binom{k-m}{i-m} \left\langle \begin{matrix} -1 \\ k-m-1 \end{matrix} \right\rangle.$$

Proof. From

$$\left\langle \begin{matrix} i \\ k \end{matrix} \right\rangle = \sum_{m=1}^i c_m(k) \binom{i}{m}, \quad i \leq k-1,$$

we deduce by inversion

$$c_i(k) = \sum_{m=1}^i (-1)^{i-m} \binom{i}{m} \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle, \quad i \leq k-1,$$

which is a direct consequence of the classical identity

$$\sum_{m=p}^i (-1)^{i-m} \binom{i}{m} \binom{m}{p} = \delta_{ip}.$$

Therefore we have

$$\begin{aligned} \frac{1}{1-t} \binom{k}{i} c_i(k) &= \sum_{m=1}^i (-1)^{i-m} \frac{1}{1-t} \binom{k}{i} \binom{i}{m} \left\langle \frac{m}{k} \right\rangle \\ &= \sum_{m=1}^i (-1)^{i-m} \binom{k-m}{i-m} \frac{1}{1-t} \binom{k}{m} \left\langle \frac{m}{k} \right\rangle. \end{aligned}$$

We apply (4.1). □

The first values are given by

$$\begin{aligned} \frac{1}{1-t} k c_1(k) &= \left\langle \frac{-1}{k-2} \right\rangle, \\ \frac{1}{1-t} \binom{k}{2} c_2(k) &= t \left\langle \frac{-1}{k-3} \right\rangle - (k-1) \left\langle \frac{-1}{k-2} \right\rangle, \\ \frac{1}{1-t} \binom{k}{3} c_3(k) &= (t^2 + \frac{k-1}{2}(1-t)) \left\langle \frac{-1}{k-4} \right\rangle - t(k-2) \left\langle \frac{-1}{k-3} \right\rangle + \binom{k-1}{2} \left\langle \frac{-1}{k-2} \right\rangle, \\ \frac{1}{1-t} \binom{k}{4} c_4(k) &= t(t^2 + \frac{5k-8}{6}(1-t)) \left\langle \frac{-1}{k-5} \right\rangle - (k-3)(t^2 + \frac{k-1}{2}(1-t)) \left\langle \frac{-1}{k-4} \right\rangle \\ &\quad + t \binom{k-2}{2} \left\langle \frac{-1}{k-3} \right\rangle - \binom{k-1}{3} \left\langle \frac{-1}{k-2} \right\rangle. \end{aligned}$$

5 An application

A partition $\rho = (\rho_1, \dots, \rho_r)$ is a finite weakly decreasing sequence of nonnegative integers, called parts. The number $l(\rho)$ of positive parts is called the length of ρ , and $|\rho| = \sum_{i=1}^r \rho_i$ the weight of ρ .

For any partition ρ and any integer $1 \leq i \leq l(\rho) + 1$, we denote by $\rho^{(i)}$ the partition μ (if it exists) such that $\mu_j = \rho_j$ for $j \neq i$ and $\mu_i = \rho_i + 1$. Similarly for any integer $1 \leq i \leq l(\rho)$, we denote by $\rho_{(i)}$ the partition ν (if it exists) such that $\nu_j = \rho_j$ for $j \neq i$ and $\nu_i = \rho_i - 1$.

In the study of the symmetric groups [4], the following differential system is encountered. To any partition ρ we associate a function $\psi_\rho(u)$ with the conditions

$$\begin{aligned} \sum_{i=1}^{l(\rho)+1} \frac{d}{du} \psi_{\rho^{(i)}}(u) &= \sum_{i=1}^{l(\rho)+1} (\rho_i - i + 1) \psi_{\rho^{(i)}}(u), \\ \sum_{i=1}^{l(\rho)+1} (\rho_i - i + 1) \frac{d}{du} \psi_{\rho^{(i)}}(u) &= \sum_{i=1}^{l(\rho)+1} (\rho_i - i + 1)^2 \psi_{\rho^{(i)}}(u) + t|\rho| \psi_\rho(u) - (1-t) \sum_{i=1}^{l(\rho)} \psi_{\rho_{(i)}}(u). \end{aligned}$$

This first order (overdetermined) differential system must be solved with the initial conditions $\psi_\rho(0) = 0$.

In this section we shall only consider the elementary case where $\rho = (r, 1^s)$ is a hook partition. In this situation the differential system becomes

$$\frac{d}{du} \left(\psi_{r+1,1^s}(u) + \psi_{r,2,1^{s-1}}(u) + \psi_{r,1^{s+1}}(u) \right) = r\psi_{r+1,1^s}(u) - (s+1)\psi_{r,1^{s+1}}(u), \quad (5.1)$$

$$\begin{aligned} \frac{d}{du} \left(r\psi_{r+1,1^s}(u) - (s+1)\psi_{r,1^{s+1}}(u) \right) &= r^2\psi_{r+1,1^s}(u) + (s+1)^2\psi_{r,1^{s+1}}(u) \\ &+ t(r+s)\psi_{r,1^s}(u) - (1-t)\left(\psi_{r-1,1^s}(u) + \psi_{r,1^{s-1}}(u)\right). \end{aligned} \quad (5.2)$$

The reader may check that for $t = 1$ the solutions $\psi_{r,1^s}(u)$ are given by

$$\begin{aligned} (r+s)! \psi_{r,1^s}(u) &= (e^u - 1)^{r-1} (e^{-u} - 1)^s \\ &= \sum_{i=-s}^{r-1} (-1)^{r+s+i-1} \binom{r+s-1}{r-i-1} e^{iu}. \end{aligned}$$

For t arbitrary, the following partial results give an idea about the high complexity of this problem.

Let us restrict to the most elementary situation $s = 0$. By linear combination, the differential system (5.1)-(5.2) is easily transformed to

$$(r+1) \psi'_{r+1} = r(r+1) \psi_{r+1} + tr\psi_r - (1-t)\psi_{r-1}, \quad (5.3)$$

$$(r+1) \psi'_{r,1} = -(r+1) \psi_{r,1} - tr\psi_r + (1-t)\psi_{r-1}, \quad (5.4)$$

which must be solved with the initial conditions $\psi_r(0) = \psi_{r,1}(0) = 0$.

Proposition 2. *We have*

$$(-1)^r r! \psi_r(u) = \sum_{i=1}^{r-1} \left\langle \begin{matrix} r-1 \\ r-i-1 \end{matrix} \right\rangle \left\langle \begin{matrix} -1 \\ i-1 \end{matrix} \right\rangle e^{iu} - \left\langle \begin{matrix} r-2 \\ r-2 \end{matrix} \right\rangle.$$

Proof. The statement is easily checked for $r \leq 3$ since we have

$$\psi_1(u) = 0, \quad 2\psi_2(u) = e^u - 1, \quad -6\psi_3(u) = t(-e^{2u} + 2e^u - 1).$$

Inspired by the $t = 1$ case, we may look for a solution of (5.3) under the form

$$(-1)^r r! \psi_r(u) = \sum_{i=0}^{r-1} a_i^{(r)} e^{iu}.$$

By identification of the coefficients of exponentials, we obtain

$$\frac{r-i-1}{r-1} a_i^{(r)} = t a_i^{(r-1)} + (1-t) a_i^{(r-2)}.$$

For $1 \leq i \leq r-2$, by induction on r this relation yields

$$a_i^{(r)} = \left\langle \begin{matrix} r-1 \\ r-i-1 \end{matrix} \right\rangle \left\langle \begin{matrix} -1 \\ i-1 \end{matrix} \right\rangle.$$

For $i=0$ by induction on r we get similarly

$$a_0^{(r)} = -\left\langle \begin{matrix} r-2 \\ r-2 \end{matrix} \right\rangle.$$

For $i=r-1$ the value of $a_{r-1}^{(r)}$ is not defined. But the latter may be obtained from the initial condition

$$\psi_r(0) = \sum_{i=0}^{r-1} a_i^{(r)} = 0.$$

Actually applying the generalized Chu-Vandermonde formula of Corollary 3, written with $x=-1$, $y=r-1$ and $k=r-2$, we have

$$\begin{aligned} -a_{r-1}^{(r)} &= \sum_{i=1}^{r-2} \left\langle \begin{matrix} r-1 \\ r-i-1 \end{matrix} \right\rangle \left\langle \begin{matrix} -1 \\ i-1 \end{matrix} \right\rangle - \left\langle \begin{matrix} r-2 \\ r-2 \end{matrix} \right\rangle \\ &= \sum_{i=0}^{r-3} \left\langle \begin{matrix} r-1 \\ r-i-2 \end{matrix} \right\rangle \left\langle \begin{matrix} -1 \\ i \end{matrix} \right\rangle - \sum_{i=0}^{r-2} \left\langle \begin{matrix} r-1 \\ r-i-2 \end{matrix} \right\rangle \left\langle \begin{matrix} -1 \\ i \end{matrix} \right\rangle \\ &= -\left\langle \begin{matrix} -1 \\ r-2 \end{matrix} \right\rangle. \end{aligned}$$

□

Proposition 3. *We have*

$$(-1)^{r+1}(r+1)! \psi_{r,1}(u) = \sum_{i=1}^{r-1} \frac{r-i}{i+1} \left\langle \begin{matrix} r \\ r-i \end{matrix} \right\rangle \left\langle \begin{matrix} -1 \\ i-1 \end{matrix} \right\rangle e^{iu} - r \left\langle \begin{matrix} r-1 \\ r-1 \end{matrix} \right\rangle + \frac{r+1}{1-t} \left\langle \begin{matrix} r \\ r+1 \end{matrix} \right\rangle e^{-u}.$$

Proof. It is similar to the previous one. Inspired by the $t=1$ case, we look for a solution of (5.4) under the form

$$(-1)^{r+1}(r+1)! \psi_{r,1}(u) = \sum_{i=-1}^{r-1} b_i^{(r)} e^{iu}.$$

By identification of the coefficients of exponentials, we obtain

$$(i+1)b_i^{(r)} = r(ta_i^{(r)} + (1-t)a_i^{(r-1)}).$$

For $1 \leq i \leq r-1$ it yields

$$\begin{aligned} (i+1)b_i^{(r)} &= r \left(t \left\langle \begin{matrix} r-1 \\ r-i-1 \end{matrix} \right\rangle + (1-t) \left\langle \begin{matrix} r-2 \\ r-i-2 \end{matrix} \right\rangle \right) \left\langle \begin{matrix} -1 \\ i-1 \end{matrix} \right\rangle \\ &= (r-i) \left\langle \begin{matrix} r \\ r-i \end{matrix} \right\rangle \left\langle \begin{matrix} -1 \\ i-1 \end{matrix} \right\rangle. \end{aligned}$$

Similarly for $i = 0$ we get

$$b_0^{(r)} = -r \left(t \left\langle \begin{smallmatrix} r-2 \\ r-2 \end{smallmatrix} \right\rangle + (1-t) \left\langle \begin{smallmatrix} r-3 \\ r-3 \end{smallmatrix} \right\rangle \right) = -r \left\langle \begin{smallmatrix} r-1 \\ r-1 \end{smallmatrix} \right\rangle.$$

For $i = -1$ the value of $b_{-1}^{(r)}$ is not defined. But we may obtain

$$b_{-1}^{(r)} = \frac{r+1}{1-t} \left\langle \begin{smallmatrix} r \\ r+1 \end{smallmatrix} \right\rangle$$

from the initial condition

$$\psi_{r,1}(0) = \sum_{i=-1}^{r-1} b_i^{(r)} = 0.$$

Actually applying Corollary 3 and Proposition 1, we have

$$\begin{aligned} -b_{-1}^{(r)} &= \sum_{i=1}^{r-1} \frac{r-i}{i+1} \left\langle \begin{smallmatrix} r \\ r-i \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} -1 \\ i-1 \end{smallmatrix} \right\rangle - r \left\langle \begin{smallmatrix} r-1 \\ r-1 \end{smallmatrix} \right\rangle \\ &= \sum_{i=1}^r \frac{r(i+1) - (r+1)i}{i+1} \left\langle \begin{smallmatrix} r \\ r-i \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} -1 \\ i-1 \end{smallmatrix} \right\rangle - r \left\langle \begin{smallmatrix} r-1 \\ r-1 \end{smallmatrix} \right\rangle \\ &= \sum_{i=0}^r \frac{-(r+1)i}{i+1} \left\langle \begin{smallmatrix} r \\ r-i \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} -1 \\ i-1 \end{smallmatrix} \right\rangle = -\frac{r+1}{1-t} \left\langle \begin{smallmatrix} r \\ r+1 \end{smallmatrix} \right\rangle. \end{aligned}$$

□

Unfortunately the situation becomes quickly very messy and a general formula for $\psi_{r,1^s}(u)$ is as yet unknown. The following case is obtained by putting $s = 1$ in (5.2), which leads to define $\psi_{r,1^2}(u)$ by

$$\begin{aligned} r\psi'_{r+1,1}(u) - 2\psi'_{r,1^2}(u) &= r^2\psi_{r+1,1}(u) + 4\psi_{r,1^2}(u) \\ &\quad + t(r+1)\psi_{r,1}(u) - (1-t)(\psi_{r-1,1}(u) + \psi_r(u)), \end{aligned}$$

with the initial condition $\psi_{r,1^2}(0) = 0$. Inspired by the $t = 1$ case, we look for a solution under the form

$$(-1)^r(r+2)! \psi_{r,1^2}(u) = \sum_{i=-2}^{r-1} c_i^{(r)} e^{iu},$$

and we obtain the recurrence relation

$$-2(i+2)c_i^{(r)} = r(r-i)b_i^{(r+1)} - (r+1)(r+2)(tb_i^{(r)} + (1-t)(b_i^{(r-1)} + a_i^{(r)})).$$

For instance

$$\begin{aligned} -4c_0^{(r)} &= r^2b_0^{(r+1)} - (r+1)(r+2)(tb_0^{(r)} + (1-t)(b_0^{(r-1)} + a_0^{(r)})) \\ &= -r^2(r+1) \left\langle \begin{smallmatrix} r \\ r \end{smallmatrix} \right\rangle + (r+1)(r+2) \left(tr \left\langle \begin{smallmatrix} r-1 \\ r-1 \end{smallmatrix} \right\rangle + (1-t)((r-1)+1) \left\langle \begin{smallmatrix} r-2 \\ r-2 \end{smallmatrix} \right\rangle \right) \\ &= 2r(r+1) \left\langle \begin{smallmatrix} r \\ r \end{smallmatrix} \right\rangle. \end{aligned}$$

The reader may check that the solutions are given by

$$\begin{aligned}
& (-1)^r (r+2)! \psi_{r,1^2}(u) = \\
& \sum_{i=1}^{r-1} \left(\frac{(r-i)(r-i+1)}{(i+1)(i+2)} \left\langle \begin{matrix} r+1 \\ r-i+1 \end{matrix} \right\rangle - (1-t) \frac{i(r+1)(r+2)}{2(i+1)(i+2)} \left\langle \begin{matrix} r-1 \\ r-i-1 \end{matrix} \right\rangle \right) \left\langle \begin{matrix} -1 \\ i-1 \end{matrix} \right\rangle e^{iu} \\
& - \binom{r+1}{2} \left\langle \begin{matrix} r \\ r \end{matrix} \right\rangle + \binom{r+2}{2} \left(\frac{2e^{-u}}{1-t} \left\langle \begin{matrix} r+1 \\ r+2 \end{matrix} \right\rangle - \left\langle \begin{matrix} r-1 \\ r \end{matrix} \right\rangle e^{-u} + \frac{e^{-2u}}{1-t} \left\langle \begin{matrix} r \\ r+2 \end{matrix} \right\rangle \right).
\end{aligned}$$

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References

- [1] W. Chu, *Elementary proofs for convolution identities of Abel and Hagen-Rothe*, Electron. J. Combin. **17** (2010), Article 24.
- [2] V. J. W. Guo, *Bijjective proofs of Gould's and Rothe's identities*, Discrete Math. **308** (2008), 1756–1759.
- [3] S. J. X. Hou, J. Zeng, *Two new families of q -positive integers*, Ramanujan J. **14** (2007), 421–435.
- [4] M. Lassalle, *Class expansion of some symmetric functions in Jucys-Murphy elements*, J. Alg. (2013), to appear.
- [5] M. Lassalle, *A new family of positive integers*, Ann. Comb. **6** (2002), 399–405.